Circle Theorems

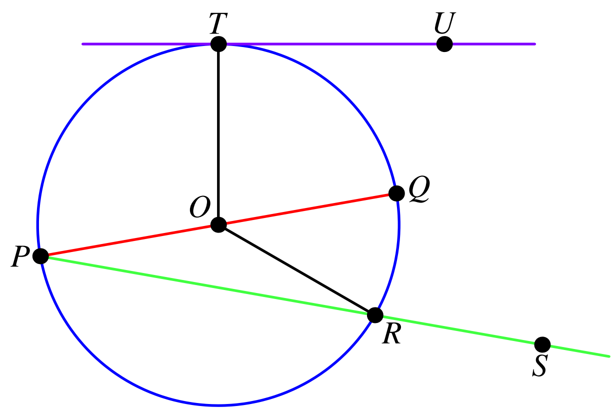
PSIA Jr. High Mathematics Special Topic for Spring 2026 and Spring 2027 is Circle Theorems. This topic will explore the relationships between segments in and around a circle as well as angles and arcs. The ideas presents range from fundamental definitions to advanced geometric concepts. Students are encouraged to develop proofs for the theorems presented in this paper, using basic assumptions and definitions from geometry.

Definitions

A **circle** is defined as the set of points that are a given distance r from a fixed point O. The point O is called the **center**. The distance r is the **radius** of the circle. Additionally, the segment from O to any point P on the circle is called a **radius** (*pl. radii*).

The segment joining two distinct points P and Q on the circle is called a **chord**. A chord that passes through the center of the circle is called a **diameter**. A **tangent** is a line in the plane of a circle that intersects the circle in exactly one point. The point is called the **point of tangency**. A **secant** is a line that contains a chord.

radii	$\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, $\overline{OT}$
diameter	$\overline{PQ}$
chord	$\overline{PQ}$, $\overline{PR}$
tangent	$\overline{TU}$
secant	$\overline{SP}$



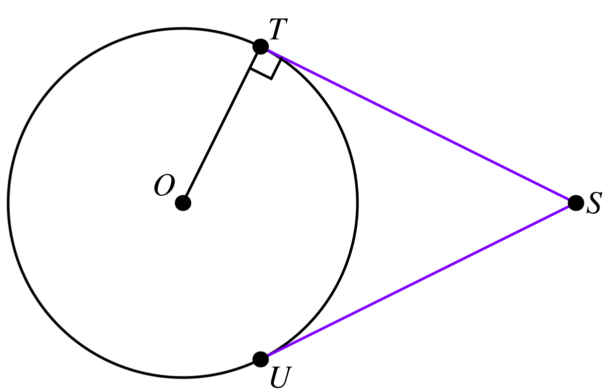
Tangent Theorems

Theorem 1

A tangent line is perpendicular to the radius drawn to the point of tangency.

Theorem 2

Tangents to a circle from a point are congruent.



In circle O, points T and U are the points of tangency from point S. Since tangent lines are perpendicular to the radius at that point, $\overline{OT} \perp \overline{ST}$. Additionally, the two tangents from a fixed point to a circle are congruent. Thus, $\overline{ST} \cong \overline{SU}$.

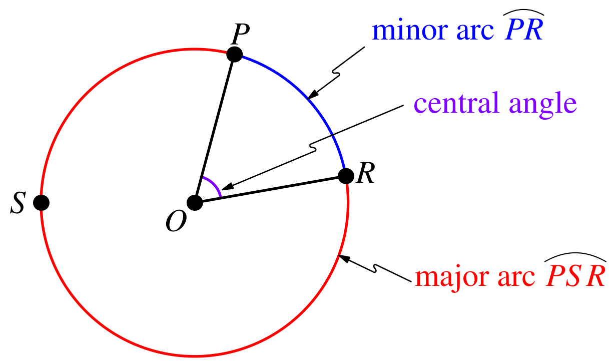
Example 1

In circle O, point T is a point of tangency from point P. If $OT = 6$ and $PT = 8$, find OP.

Solution:
Since T is a point of tangency and O is the center of the circle, OT is perpendicular to PT. This forms a right triangle with OP as the hypotenuse. Using the Pythagorean Theorem, we have $(OT)^2+(PT)^2 = (OP)^2$. Using $OT = 6$ and $PT = 8$, we have $(OP)^2 = 6^2 + 8^2 = 36 + 64 = 100$, which gives $OP = 10$.

Arcs and Angles

A **central angle** is an angle with its vertex at the center of a circle.
A **minor arc** is formed by part of the circle that is interior to the angle that defines the arc. The other part of the circle is the **major arc**. Minor arcs are notated with the two endpoints of the arc; major arcs are notated using three points on the circle (two endpoints and one extra point on the major arc to signify the difference between the two possible arcs).



The **measure of a minor arc** is equal to its defining central angle.
The **measure of a major arc** is 360° minus the measure of the corresponding minor arc.
The measure of a semicircular arc is 180°.

Example 2

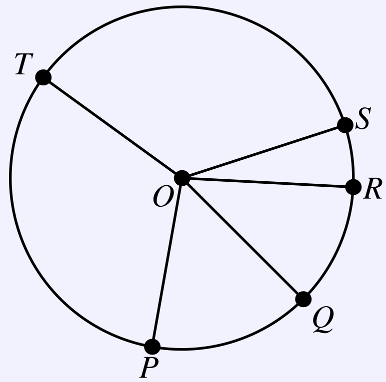
Points A, B, and Q lie on circle C and $m\angle ACB = 80°$. Find $m\widehat{AB}$ and $m\widehat{AQB}$.

Solution: The measure of the minor arc AB is equal to the measure of the central angle $\angle ACB = 80°$. The measure of the major arc AQB is 360° minus the measure of the minor arc.

$$m\widehat{AQB} = 360° - m\widehat{AB} = 360° - 80° = 280°.$$

Example 3

In circle O, $m\overset{\frown}{STP} = 242°$, $m\overset{\frown}{QS} = 63°$, and $m\overset{\frown}{PR} = 97°$. Find $m\overset{\frown}{QR}$.



Solution: First, compute the measure of the minor arc PS: $360° - m\overset{\frown}{STP} = 360° - 242° = 118°$. Notice that $m\overset{\frown}{PS} = m\overset{\frown}{PQ} + m\overset{\frown}{QR} + m\overset{\frown}{RS} = 118°$. We also know that $m\overset{\frown}{QS} = m\overset{\frown}{QR} + m\overset{\frown}{RS} = 63°$ and $m\overset{\frown}{PR} = m\overset{\frown}{PQ} + m\overset{\frown}{QR} = 97°$. If we add these equations together, we get

$$\begin{aligned}
(m\overset{\frown}{QR} + m\overset{\frown}{RS}) + (m\overset{\frown}{PQ} + m\overset{\frown}{QR}) &= 63° + 97° \\
m\overset{\frown}{QR} + (m\overset{\frown}{PQ} + m\overset{\frown}{QR} + m\overset{\frown}{RS}) &= 160° \\
m\overset{\frown}{QR} + 118° &= 160° \\
m\overset{\frown}{QR} &= 42°
\end{aligned}$$

Inscribed Angles

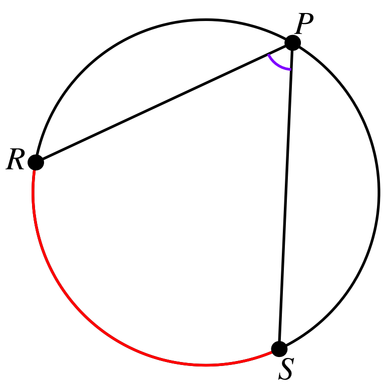
An **inscribed angle** is an angle whose vertex is on a circle and whose sides are chords of the circle.

Theorem 3

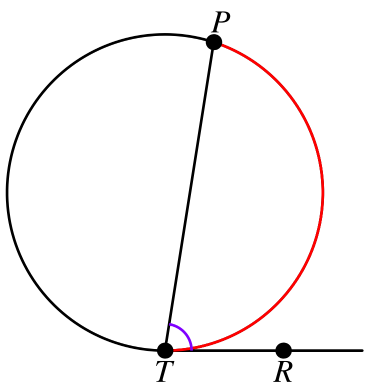
The measure of an inscribed angle is half of its intercepted arc.

Theorem 4

The measure of an angle formed by a chord and a tangent is equal to half the measure of the intercepted arc.



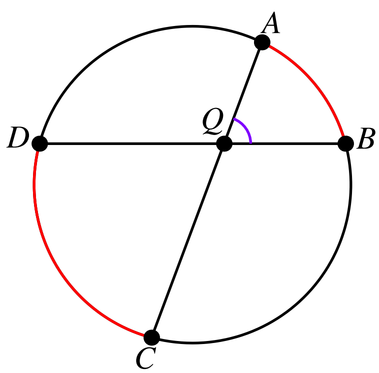
$\angle RPS$ is inscribed in the circle
$m\angle RPS = \frac{1}{2}m\widehat{RS}$



$\overline{TR}$ is tangent to the circle
$m\angle PTR = \frac{1}{2}m\widehat{PT}$

Theorem 5

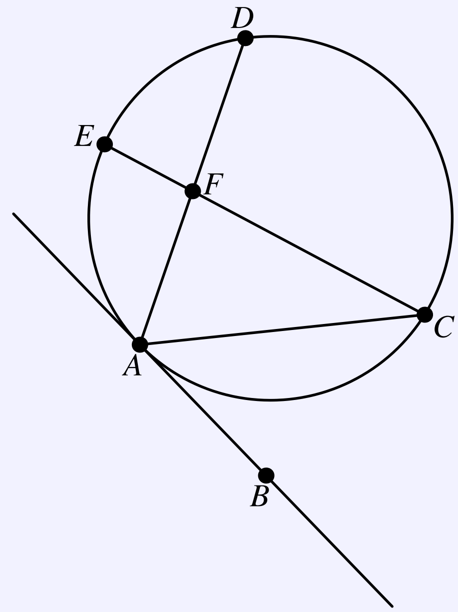
The measure of an angle formed by two chords that intersect inside a circle is equal to half the sum of the measures of the intercepted arcs.



$m\angle AQB = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$

Example 4

In the circle, $m\widehat{AC} = 104°$, $m\angle DAC = 65°$, and $m\widehat{EA} = 68°$. Also, $\overline{AB}$ is tangent to the circle at point A. Find the following: $m\angle CAB$, $m\widehat{CD}$, $m\angle ACE$, $m\widehat{DE}$.



Solution: $\angle CAB$ is the angle formed by a tangent line and a chord. The angle is half the size of the intercepted arc: $m\angle CAB = \frac{1}{2}m\widehat{AC} = \frac{1}{2}(104°) = 52°$.

To find $m\widehat{CD}$, use the inscribed angle DAC. The inscribed angle is half the size of the intercepted arc: $m\widehat{CD} = 2 \cdot m\angle DAC = 2(65°) = 130°$

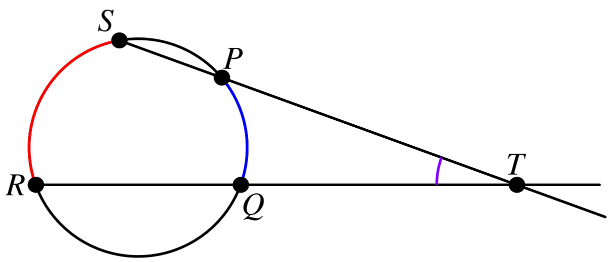
Similarly, to find $m\angle ACE$, use the minor intercepted arc EA: $m\angle ACE = \frac{1}{2}m\widehat{EA} = \frac{1}{2}(68°) = 34°$.

Finally, to find $m\widehat{DE}$, there are several ways to go at this point. One method is to find $m\angle AFC$. Since AFC is a triangle, the sum of the angles is 180°. We already know two of these angles: 65° and 34°. This gives $m\angle AFC = 180° - 65° - 34° = 81°$. Then, this angle is the average of the two arcs that is formed by chords around it. $m\angle AFC = \frac{1}{2}(m\widehat{DE} + m\widehat{AC})$. Solving this for $m\widehat{DE}$ gives $m\widehat{DE} = 2 \cdot m\angle AFC - m\widehat{AC} = 2(81°) - 104° = 58°$.

Can you find another way to solve for $m\widehat{DE}$? Hint: Create a point X on $\overleftrightarrow{AB}$, outside the segment AB on the A-side.

Theorem 6

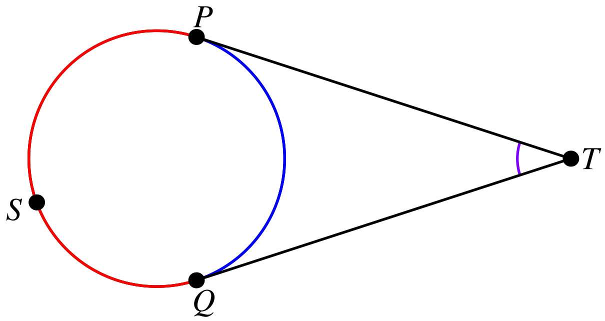
The measure of an angle formed by two secants is equal to half the difference of the measures of the intercepted arcs.



$$m\angle PTQ = \frac{1}{2}(m\widehat{RS} - m\widehat{PQ})$$

Theorem 7

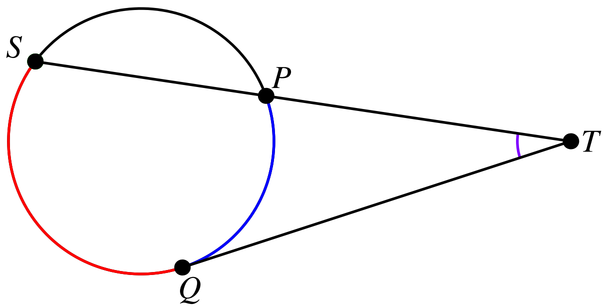
The measure of an angle formed by two tangents is equal to half the difference of the measures of the intercepted arcs.



$$m\angle PTQ = \frac{1}{2}(m\widehat{PSQ} - m\widehat{QP})$$

Theorem 8

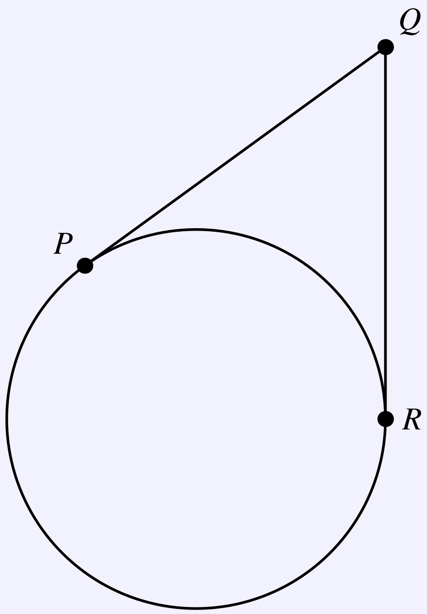
The measure of an angle formed by a secant and a tangent is equal to half the difference of the measures of the intercepted arcs.



$$m\angle PTQ = \frac{1}{2}(m\widehat{QS} - m\widehat{QP})$$

Notice that although these theorems are written as three separate ideas, the formula is the same: find the difference between the intercepted arcs and divide by 2.

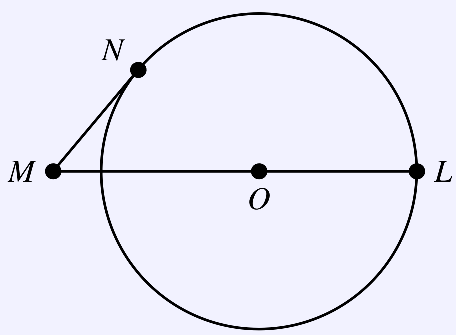
Example 5



In the figure, $\overline{QP}$ and $\overline{QR}$ are tangent to the circle and $m\widehat{PR} = 126°$. Find $m\angle PQR$.

Solution: Let S be a point on the major arc $\widehat{PR}$. Since $m\widehat{PR} = 126°$, $m\widehat{PSR} = 360° - 126° = 234°$. Then, $m\angle PQR = \frac{1}{2}(m\widehat{PSR} - m\widehat{PR}) = \frac{1}{2}(234° - 126°) = 54°$.

Example 6



With circle O, $\overline{MN}$ is a tangent. If $m\angle LMN = 50°$, find $m\widehat{LN}$.

Solution: Since secant $\overline{ML}$ passes through the center, we know the secant contains the diameter of the circle. This gives an arc measuring 180°. Let P be the point on $\overline{ML}$ intersecting the circle opposite L. Using this fact, we have $m\widehat{NP} + m\widehat{LN} = 180°$, or $m\widehat{NP} = 180° - m\widehat{LN}$.

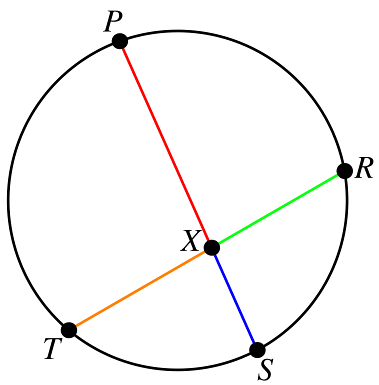
Using the difference formula, we have

$$\begin{aligned}
50° = m\angle LMN &= \frac{1}{2}(m\widehat{LN} - m\widehat{NP}) \\
&= \frac{1}{2}(m\widehat{LN} - (180° - m\widehat{LN})) \\
&= \frac{1}{2}(2 \cdot m\widehat{LN} - 180°) \\
&= m\widehat{LN} - 90° \\
m\widehat{LN} &= 140°
\end{aligned}$$

Chords and Segments

Theorem 9

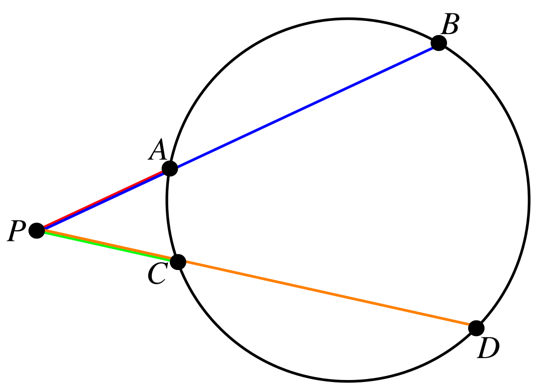
When two chords intersect inside a circle, the product of the segments of one chord equals the product of the segments of the other chord.



$$(PX)\cdot(XS) = (RX)\cdot(XT)$$

Theorem 10

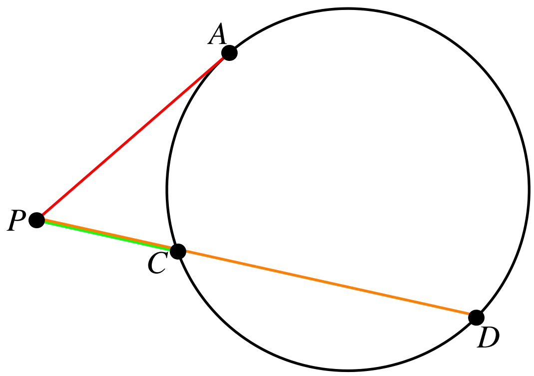
When two secant segments are drawn to a circle from a point outside the circle, the product of one secant segment and its external segment equals the product of the other secant segment and its external segment.



$$(PA)\cdot(PB) = (PC)\cdot(PD)$$

Theorem 11

When a secant and a tangent are drawn to a circle from a point outside the circle, the product of one secant segment and its external segment equals the square of the tangent.



$$(PA)^2 = (PC)\cdot(PD)$$

Can you see how to derive the formula for Theorem 11 from the formula for Theorem 10? What has to happen to point B in Thm 10 to get Thm 11?

Ptolemy's Theorem

A polygon is said to be **cyclic** if all vertices of the polygon lie on a common circle.

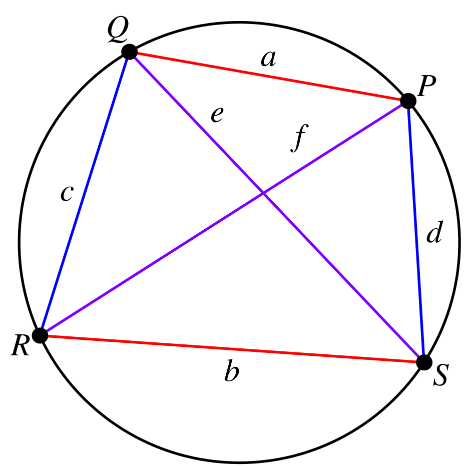
The angles across from each other in a cyclic quadrilateral are supplementary.

Ptolemy's Theorem

If a quadrilateral is cyclic, then the product of the lengths of the diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.

Converse of Ptolemy's Theorem

In a quadrilateral, if the sum of the products of the lengths of its two pairs of opposite sides is equal to the product of the lengths of its diagonals, then the quadrilateral is cyclic.



$$(a \cdot b) + (c \cdot d) = e \cdot f$$

Example 7

Quadrilateral $ABCD$ is inscribed in a circle, with $AB = 8$, $BC = 9$, $CD = 3$, and $AD = 10$. Find the product of AC and BD.

Solution: Since the quadrilateral is cyclic, we can use Ptolemy's Theorem.

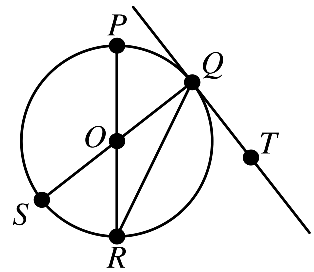
$$\begin{aligned} AC \cdot BD &= AB \cdot CD + AD \cdot BC \\ &= (8)(3) + (10)(9) \\ &= 24 + 90 \\ &= 114 \end{aligned}$$

Challenge: What do you get from Ptolemy's Theorem if the quadrilateral inscribed in the circle is a rectangle?

Practice Questions

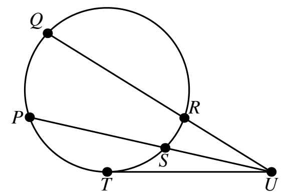
Given $\overline{PR}$ and $\overline{QS}$ are diameters of circle O, $\overline{QT}$ is tangent to the circle, and $m\angle POQ = 52°$.

1. Find $m\angle ROQ$.
2. Find $m\angle OQR$.
3. Find $m\widehat{RS}$.
4. Find $m\widehat{QRP}$.
5. Find $m\angle RQT$.



Given $\overline{TU}$ is tangent to the circle, $m\widehat{PT} = 72°$, $m\widehat{PQ} = 88°$, $m\widehat{RS} = 24°$, and $m\widehat{ST} = 44°$.

6. Find $m\angle TUS$.
7. Find $m\angle SUR$.



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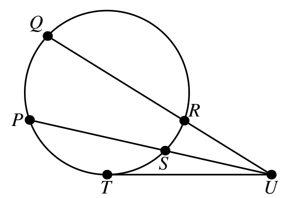
Given $\overline{TU}$ is tangent to the circle.

8. If $TU = 8$ and $US = 6$, find PS.
9. If $US = 12$, $PS = 20$, and $QR = 40$, find UR.



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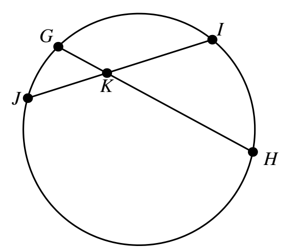
10. If $GK = 4$, $IK = 6$, and $JK = 8$, find HK.



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11. A rectangle is inscribed in a circle of radius 5. One side of the rectangle is 6. What is the other side?
12. A quadrilateral with sides 5, 8, 3, and 9 (going around the circle) is inscribed in a circle. Find the product of its diagonals.

Answers: **1.** 128° **2.** 26° **3.** 52° **4.** 308° **5.** 64° **6.** 14° **7.** 32° **8.** 14/3 **9.** 8 **10.** 12 **11.** 8 **12.** 87