Principles of Counting

PSIA Jr. High Mathematics Special Topics for Spring 2024 and Spring 2025 is Principles of Counting. This topic includes the fundamentals of the field of mathematics called *combinatorics*. Combinatorics is central to many areas of study, including computer science, data science, probability theory, abstract algebra, cryptography, and more! This article aims to teach PSIA students the basics of combinatorics at an accessible level, encourage appreciation for this area of mathematics, and provide enrichment activities and achievement standards. I hope you will enjoy this subject matter!

Multiplication Principle

The band director is the head of a marching band. The school board would like to know how many different uniforms are possible for the band. The director has 3 different shirts band members can wear and 2 different pants they can wear. All shirts match all pants. How many different uniforms can be created?

To solve this, let's create the uniforms. We can say that the shirts are white, purple, and gold and that the pants are khaki and black. Here are the combinations of shirts and pants together:

Shirts \ Pants	Khaki	Black
White	White-Khaki	White-Black
Purple	Purple-Khaki	Purple-Black
Gold	Gold-Khaki	Gold-Black

The director has 6 different uniforms. It was easy to count them out because the numbers were so small. But what if the numbers were larger? What if you had 18 different shirts and 15 different pants? It would take too long to list all combinations. Luckily, there is an easier way to do the counting. We can use the Multiplication Principle.

Multiplication Principle

If there are a ways to make one selection and b ways to make a different selection, then there are $a \times b$ ways to make both selections together.

The Multiplication Principle says that when you are making two different type of choices, you can just multiply the number of choices for each together to get the total number of all choices.

Let's see how to use the Multiplication Principle to solve this problem.

Example 1

There are 3 choices for shirts and 2 choices for pants. Shirts are different from pants (obviously–that realization will make more sense later). Multiply the number of shirts by the number of pants to get the number of uniforms.

$$\underset{\text{shirts}}{\boxed{3}} \times \underset{\text{pants}}{\boxed{2}} = 6$$

But wait! The director realizes that he must also consider the different colors of gloves and different colors of shoes. Now, he has four factors to take into account: shirts, pants, gloves, and shoes. Luckily, the Multiple Principle can be extended to any number of factors.

Extended Multiplication Principle

Suppose there are n separate factors to consider and call them $A_1, A_2, A_3, \ldots, A_n$. If there are a_1 ways to make a selection for A_1, a_2 ways to make a selection for A_2, …, and a_n ways to make a selection for A_n, then the total number of ways to make all selections is

$$a_1 \times a_2 \times a_3 \times \cdots \times a_n.$$

Example 2

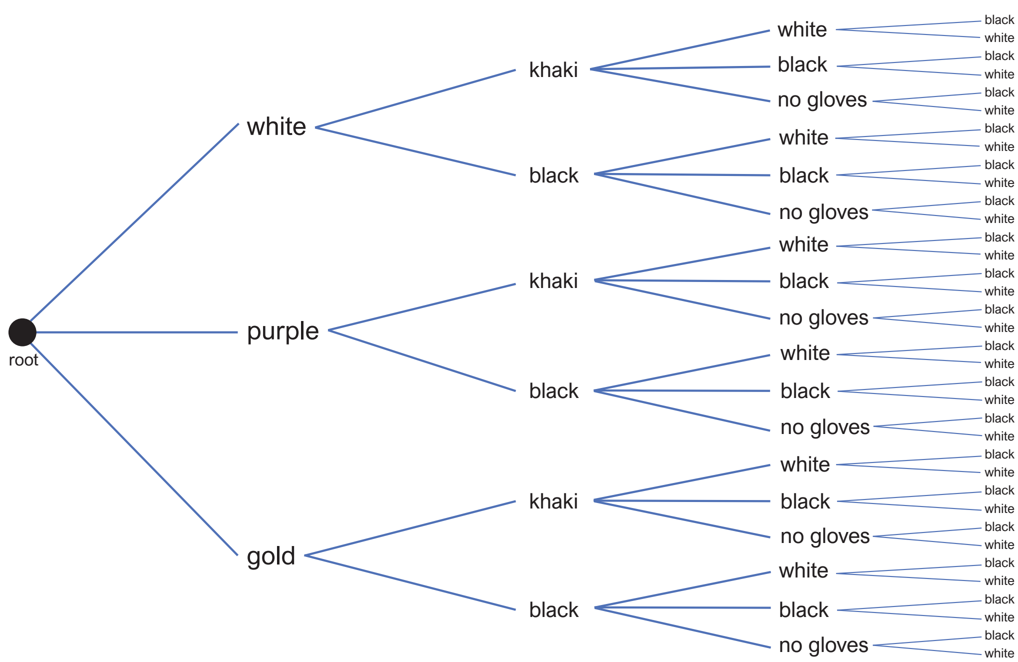
The band director has the original 3 shirts and 2 pants, as before. But now, he also has 3 different types of gloves, and 2 types of shoes. How many total uniforms are possible now?

Solution:

$$\underset{\text{shirts}}{\boxed{3}} \times \underset{\text{pants}}{\boxed{2}} \times \underset{\text{gloves}}{\boxed{3}} \times \underset{\text{shoes}}{\boxed{2}} = 36$$

Trees

A chart was a good way to visualize the total number of combinations when there were 2 choices: shirts and pants. But now we must factor in gloves and shoes. We can use trees to make this visualization.



Start with the "root" where the tree begins. Then, make branches for the different factors. For each branch, make further branches for the next factor, and so on.

Addition Principle

First, let's define some terms. An **event** is a set or list of results or outcomes. For example, "shirts" was one of the events from the above examples. Inside the event "shirts", we have the outcomes "white", "purple", and "gold" – the colors of the shirts. When you roll a standard 6-sided die, the event is the roll of the die and the outcomes are the possible rolls: 1, 2, 3, 4, 5, and 6. What are the other events and their outcomes used in the uniform example? *

Addition Principle

If there are a ways to select from event A and b ways to select from event B, AND events A and B cannot happen at the same time, then there are $a + b$ ways to select from the events A and B.

The key to using the Addition Principle is to notice when the events can and cannot occur at the same time. In the above examples, the events were shirts and pants. Do you pick a shirt and a pair of pants to make the uniform? Yes, so these events do occur at the same time and the Additional Principle does not apply.

Here's an example where the Addition Principle does apply.

Example 3

In the band, there are 15 seniors and 18 juniors. The director needs to pick one upperclassman (senior or junior) to be the Band President. How many different ways can the Band President be selected?

Solution: There are two possibilities for the president: a senior or a junior. Is it possible to be a senior and a junior at the same time? No. Thus, we need to use the Addition Principle:

$$\underbrace{\boxed{15}}_{\text{seniors}} + \underbrace{\boxed{18}}_{\text{juniors}} = 33$$

There are 33 different ways to choose the president.

One key way to distinguish between using the Multiplication Principle and Addition Principle is the use of the words "and" and "or". In general, if the events are connected by "and", use the Multiplication Principle. If the events are connected by "or", use the Additional Principle. The above example stated that the president needed to be a senior *or* a junior. Since seniors and juniors are completely separate, we used the Addition Principle.

There is a version of the Addition Principle for more than two events.

*pants (khaki, black); gloves (white, black, none); shoes (black, white)

Extended Addition Principle

If there are

- a_1 ways to select from event A_1,
- a_2 ways to select from event A_2,
- a_3 ways to select from event A_3, $\ldots$, and,
- a_n ways to select from event A_n

AND all events $A_1, A_2, A_3, \ldots, A_n$ cannot happen at the same time, then there are

$$a_1 + a_2 + a_3 + \ldots + a_n$$

ways to select from all events A_1 through A_n.

Example 4

The band has 2 drum majors, 18 woodwinds players, 40 brass players, 12 on-field percussionists, and 8 players in the pit (timpani, xylophones, etc.) How many ways can one person be selected for Bandmember of the Week?

Solution: All of these positions are separate. We can use the Extended Addition Principle:

$$\underset{\text{drum majors}}{\boxed{2}} + \underset{\text{woodwinds}}{\boxed{18}} + \underset{\text{brass}}{\boxed{40}} + \underset{\text{percussion}}{\boxed{12}} + \underset{\text{pit}}{\boxed{8}} = 80$$

There are 80 possibilities for Bandmember of the Week!

Making Multiple Selections

All of the examples so far have involved selecting one person or item from a group. What happens when you need to select two or more people or items from the group? How do the calculations change?

New concerns arise:

- when we make the selection, are we able to use the same item again?
- does the order in which the selection is made matter?

Let's tackle the first concern.

When you can use the same item again, then nothing changes. If you cannot use the same item again, you must withdraw that item from consideration for the next selection.

Example 5

The band director needs to select 3 woodwind players to perform at the pep rally as a group. The parts in the music are labeled melody, harmony, and bass. How many different trios are possible?

Solution: Recall that there are 18 woodwind players in the marching band (from above). It is implied that you cannot select the same woodwind player multiple times since you need three different players for the trio. Set up a selection process.

$$\underset{\text{melody}}{\boxed{}} \times \underset{\text{harmony}}{\boxed{}} \times \underset{\text{bass}}{\boxed{}} =$$

How many woodwind players do you have for the first selection? There are 18 total woodwind players. Place 18 in the first box.

$$\underset{\text{melody}}{\boxed{18}} \times \underset{\text{harmony}}{\boxed{}} \times \underset{\text{bass}}{\boxed{}} =$$

How many woodwind players do you have for the second selection? It's no longer 18 since at this point, we have made a choice of one player. That player cannot be selected again, so we remove them from the group of select from. That leaves 17 players to select from. Place 17 in the second box.

$$\underset{\text{melody}}{\boxed{18}} \times \underset{\text{harmony}}{\boxed{17}} \times \underset{\text{bass}}{\boxed{}} =$$

And what about the third selection? We only have 16 players to select from here since we eliminated another player. Place 16 in the third box. Notice that we are using the Multiplication Principle and multiplying all of these numbers to get the final result.

$$\underset{\text{melody}}{\boxed{18}} \times \underset{\text{harmony}}{\boxed{17}} \times \underset{\text{bass}}{\boxed{16}} = 4896$$

That gives a total of 4896 ways to select the players for the Woodwind Trio at the pep rally.

Example 6

The drum majors have a secret code to get into the drum major office. This code is four digits long and each digit can be any number 0 through 9. How many different secret codes are possible?

Solution: In this problem, the digits can be reused. Start with four blanks, one for each digit.

$$\underset{\text{1st}}{\boxed{}} \times \underset{\text{2nd}}{\boxed{}} \times \underset{\text{3rd}}{\boxed{}} \times \underset{\text{4th}}{\boxed{}} =$$

There are 10 digits from the numbers 0 to 9. Each digit can be used over and over. Thus, there are 10 choices for the digits for each place in the code.

$$\underset{\text{1st}}{\boxed{10}} \times \underset{\text{2nd}}{\boxed{10}} \times \underset{\text{3rd}}{\boxed{10}} \times \underset{\text{4th}}{\boxed{10}} = 10{,}000$$

There are 10,000 possible codes for the office.

Example 7

The code to open the door to the uniform closet has four digits as well, 0 through 9, but each digit must be different from the digit immediately before it. How many different combinations are possible here?

Solution: This problem is slightly different than the previous problem. The digit that follows a digit must be different. Start with four blanks again.

$$\underset{\text{1st}}{\boxed{}} \times \underset{\text{2nd}}{\boxed{}} \times \underset{\text{3rd}}{\boxed{}} \times \underset{\text{4th}}{\boxed{}} =$$

There are 10 choices for the first digit from the numbers 0 through 9 and since there is no digit before the first one, there are no restrictions on which number can be used.

$$\underset{\text{1st}}{\boxed{10}} \times \underset{\text{2nd}}{\boxed{}} \times \underset{\text{3rd}}{\boxed{}} \times \underset{\text{4th}}{\boxed{}} =$$

Next, we need to select the second digit. From the digits 0 through 9, we know that we cannot repeat the digit that was selected for the first digit. We don't know what digit was selected for the first digit, but we do know it was one of them and therefore, there are only 9 digits that we can select from for the second digit in the code.

$$\underset{\text{1st}}{\boxed{10}} \times \underset{\text{2nd}}{\boxed{9}} \times \underset{\text{3rd}}{\boxed{}} \times \underset{\text{4th}}{\boxed{}} =$$

For the third digit, you might think there are only 8 choices, but that is not correct. The only restriction is that the third digit cannot be the same as the second digit. The number that was selected for the first digit is back in play again. So, from the digits 0 through 9, there is only one number that cannot be selected (the one that was selected for the second digit). There are 9 choices available for the third digit. And, repeating the same logic for the fourth digit, there are 9 choices for it as well.

$$\underset{\text{1st}}{\boxed{10}} \times \underset{\text{2nd}}{\boxed{9}} \times \underset{\text{3rd}}{\boxed{9}} \times \underset{\text{4th}}{\boxed{9}} = 7290$$

Permutations and Combinations

Permutations and combinations are used to solve the second concern–when the number of items you are selecting from is reduced because you cannot repeat items. Here is the difference between the two:

Permutations

Permutations give the number of ways to select r items from a set of n items when the order in which the items are selected matters. The notation for permutation is ${}_nP_r$.

Combinations

Combinations give the number of ways to select r items from a set of n items when the order in which the items are selected does not matter. The notations for combinations are ${}_nC_r$ and $\binom{n}{r}$.

Deciding when to use a permutation or a combination depends on whether or not the order is important or irrelevant when the selection is made. For example, a baseball team has 12 players and the coach needs to pick a starting lineup of 9 players and the positions that they will play (pitcher, catcher, etc.). Since the coach is picking players for different positions, we would use a permutation to calculate the number of possible rosters. Assigning the same 9 players to different positions creates a different team. The number of possible different teams is ${}_{12}\mathrm{P}_9$. I would read this as "12 pick 9" although that terminology is not standard–I made it up. I use "pick" for **p**ermutations.

On the other hand, let's suppose the coach just wants to pick 9 players and does not assign their positions. Then, a combination would be used to calculate the number of possible teams. Since all players are equal on this team, the order they are selected is irrelevant. The number of possible teams is ${}_{12}\mathrm{C}_9$, pronounced "12 choose 9." I use "choose" for **c**ombinations and this terminology is standard.

Factorials

The symbol for factorial is the exclamation mark. To find the factorial of a number, multiply every whole number from 1 up to that number, inclusive.

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Factorials are used to solve questions where you cannot repeat selected items.
Here are the first few factorials:

$$\begin{aligned} 1! &= 1 \\ 2! &= 2 \times 1 = 2 \\ 3! &= 3 \times 2 \times 1 = 6 \\ 4! &= 4 \times 3 \times 2 \times 1 = 24 \\ 5! &= 5 \times 4 \times 3 \times 2 \times 1 = 120 \end{aligned}$$

Also note that 0! is defined as 1. Some people call this "the empty selection." Can you figure out why?

Factorials of used to calculate permutations and combinations. Their formulas are:

Formulas

Permutation ${}_n\mathrm{P}_r$

$$\frac{n!}{(n-r)!} = \underbrace{n \times (n-1) \times \cdots \times (n-r+1)}_{r \text{ terms}}$$

Combination ${}_n\mathrm{C}_r$

$$\frac{n!}{r! \times (n-r)!} = \frac{n \times (n-1) \times (n-2) \times \cdots (n-r+1)}{r \times (r-1) \times \cdots \times 2 \times 1}$$

The formula for permutations starts with n and then multiplies the next r terms like it's a factorial. It just doesn't go all the way down to 1 (unless $r = n$).

The formula for combinations is the same as that of permutations, except you divide by the factorial of the number of items you are choosing. Dividing by $r!$ strips out the "ordering" aspect that is found in the permutations.

Example 8

The band must elect a President, Vice President, Secretary, and Treasurer. The band has 80 members. How many different ways can the band elect these people?

Solution: The first question to ask is "does the order matter?" The answer is yes. It is different for someone to be picked as the President than if they were picked as the Treasurer. Since the order matters, we must use permutations.

How many people do we have to pick from? 80
How many people are we picking? 4

Therefore, we have

$$ {}_{80}\mathrm{P}_4 = 80 \times 79 \times 78 \times 77 = 37{,}957{,}920 $$

ways to pick these band leadership positions.

Example 9

The band must elect a council of four members. The band has 80 members. How many different ways can the band council be elected?

Solution: Again, "does the order matter?" This time the answer is no. It does not matter if you are the first person on the council or the last person on the council. The council is seen as a whole unit with no distinction between its members. Thus, we will use combinations to find the number of ways the council can be chosen.

How many people do we have to choose from? 80
How many people are we choosing? 4

This gives us

$$ {}_{80}\mathrm{C}_4 = \frac{80 \times 79 \times 78 \times 77}{4 \times 3 \times 2 \times 1} = 1{,}581{,}580 $$

ways to choose the band council.

For some situations, it might be necessary to combine the various principles and formulas listed above.

Example 10

The band has 15 seniors, 18 juniors, 22 sophomores, and 25 freshmen. The band council must be composed of two students from each class. How many different councils are possible?

Solution: First we notice that we need students from each class. We'll set up boxes, one for each class and use the Multiplication Principle.

$$\underset{\text{seniors}}{\boxed{}} \times \underset{\text{juniors}}{\boxed{}} \times \underset{\text{sophomores}}{\boxed{}} \times \underset{\text{freshmen}}{\boxed{}} =$$

Next, we compute the number of ways to select the students from each class. For seniors, we have 15 students to select from. Since these students are participating in a council, the order that they are selected in does not matter. This means we must use combinations.

Seniors: $_{15}C_2 = \dfrac{15 \times 14}{2 \times 1} = 105$ Juniors: $_{18}C_2 = \dfrac{18 \times 17}{2 \times 1} = 153$

Sophomores: $_{22}C_2 = \dfrac{22 \times 21}{2 \times 1} = 231$ Freshmen: $_{25}C_2 = \dfrac{25 \times 24}{2 \times 1} = 300$

These numbers go in the boxes and we multiply to get the total number of ways the council can be selected.

$$\underset{\text{seniors}}{\boxed{105}} \times \underset{\text{juniors}}{\boxed{153}} \times \underset{\text{sophomores}}{\boxed{231}} \times \underset{\text{freshmen}}{\boxed{300}} = 1{,}113{,}304{,}500$$

[These numbers are a bit too difficult to multiply together for the PSIA test, but the ideas are sound.]

Sets and Set Notation

Fundamental to many areas of mathematics is the idea of sets. A **set** is a collection of items. Sets are written by listing all of the **elements** (the items in a set) between curly braces { }. For example, the set of whole numbers less than or equal to 5 is $S = \{1, 2, 3, 4, 5\}$.

The **universal set** is the set of all things currently under consideration. For example, if we are only considering the whole numbers less than 10, then the universal set is $U = \{1, 2, 3, \ldots, 10\}$. Notice also how you can use $\ldots$ (called *ellipsis*) to express a continuation of the pattern and/or omission of elements, to shorten the list in the set. The element 8 is still a member of the set U above, even though it is not explicitly written.

The **complement** of a set S is the set of everything in the universal set U that is *not* in set S. The notation for the complement of set S is S' (read as "S prime"). For example, suppose the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and set $S = \{1, 2, 3, 7\}$. Then, the complement of S is $S' = \{4, 5, 6, 8\}$, everything in U that is not in S.

The **union** of two sets S and T is denoted by $S \cup T$ and is the set of elements that are in set S or in set T or in both sets S and T. For example, if $S = \{1, 2, 3\}$ and $T = \{3, 4, 5\}$, then $S \cup T = \{1, 2, 3, 4, 5\}$. Notice that it does not matter that 3 is an element of both sets–it is still in the union.

The **intersection** of two sets S and T is denoted by $S \cap T$ and is the set of elements that are in both set S and T. Using the same sets above, $S = \{1, 2, 3\}$ and $T = \{3, 4, 5\}$, then $S \cap T = \{3\}$, since 3 is the only element in both sets S and T.

Suppose we have the sets $S = \{1, 3, 5\}$ and $T = \{2, 4, 6\}$. What is $S \cap T$ then? Notice that S contains odd numbers and T contains even numbers. There are no elements in common to both sets. Therefore, the intersection $S \cap T$ has no elements. There are two notations for this. One notation is just curly braces with nothing inside: $\{\ \}$. The other notation is a special symbol: $\varnothing$. This set is named the **empty set**.

Counting the number of elements in a set is an important topic in combinatorics. The notation for the number of elements in a set S is $n(S)$ (read as "number in S" or "n of S"). If $S = \{1, 3, 5, 7\}$, then $n(S) = 4$, since there are 4 elements in set S.

Question: Suppose S is a set that comes from the universal set U. What is $n(S) + n(S')$?†

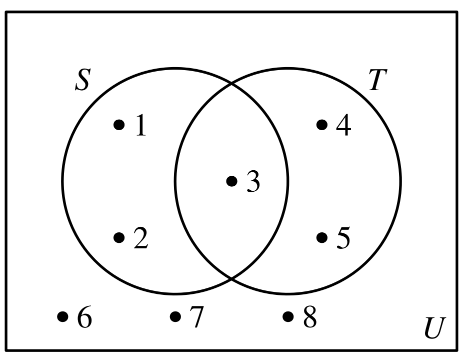
Sets can have zero elements, a finite number of elements, or an infinite number of elements.

Type	S	$n(S)$
Empty Set	$\varnothing$	0
Finite Set	$\{1, 2, 3, 4, 5\}$	5
Infinite Set	$\{2, 4, 6, 8, 10, \ldots\}$	infinitely many

Venn Diagrams

A Venn diagram is a way to visualize sets and their relationship to each other. The universal set is denoted by the box. Although sets can be represented by any shape, they are most often seen as circles.

Here is an example of the sets $S = \{1, 2, 3\}$ and $T = \{3, 4, 5\}$ from the universal set $U = \{1, 2, 3, \ldots, 8\}$.

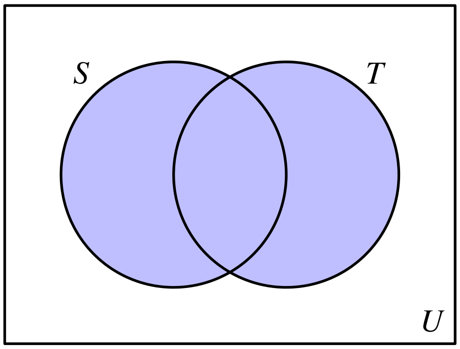


Notice how set S is surrounding the numbers 1, 2, and 3, while set T is surrounding the numbers 3, 4, and 5. The numbers 6, 7, and 8 are outside both S and T. An element that is in the set is drawn inside the circle representing the set. When an element is in both sets, the element should be drawn in the overlapping section.

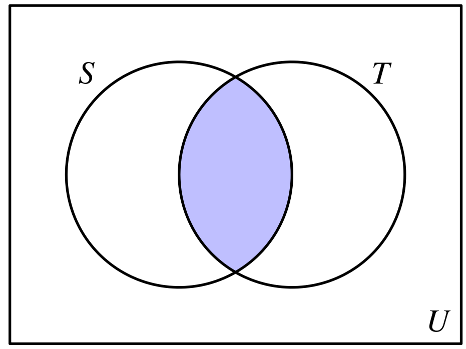
Sets can be denoted by shading in the appropriate areas of the Venn diagram. Here are some common representations.

†Answer: $n(U)$, since $n(S)$ is the number of elements in S and $n(S')$ is the number of element not in S. For elements in U, either you are in S or you are not in S. That counts all of the elements in the universal set U.

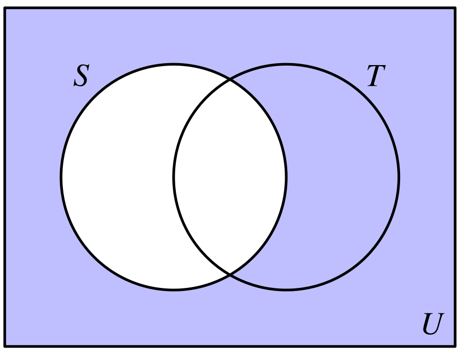
$S \cup T$



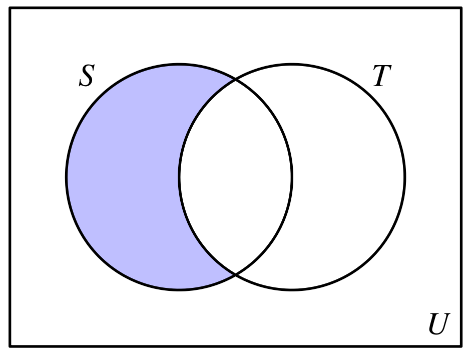
$S \cap T$



S'



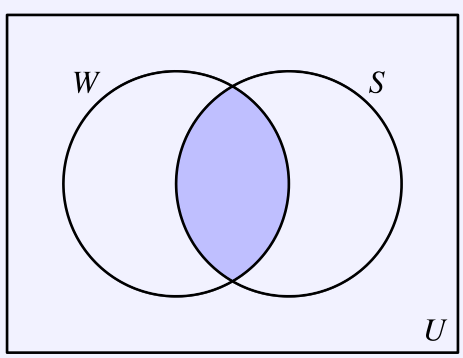
$S \cap T'$



Example 11

Let W be the set of all woodwind players and S be the set of all seniors. Knowing that some of the seniors are also woodwind players, draw the appropriate Venn diagram and shade in the area of all senior woodwind players.

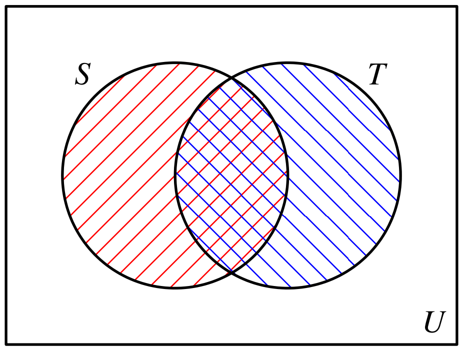
Solution: Start with a Venn diagram and draw in two overlapping sets, W for woodwinds and S for seniors. Shade in the intersection of these two sets since we are looking for bandmembers who are both woodwind players and seniors.



Inclusion-Exclusion Principle

Another use of Venn diagrams involves counting elements in each set. We just write the number of elements in each area to represent the number of elements in that area. [Notice: if you were are representing actual elements of a set, use a dot and then name of the element. If you are just writing the number of elements, just write the number without the dot.]

Notice what happens if we shade set S with lines going one way and set T with lines going the other way. What do you notice about $S \cap T$?



The set S is counted with red lines and set T is counted with blue lines. However, if you were trying to count the elements in their union, $S \cup T$, you would have counted some elements twice! The area of the diagram where their are both red and blue lines are being double counted.

To count the number of elements in $S \cup T$, you add up the number of elements in S plus the number of elements in T, and then subtract off the number of elements you double-counted, the ones in $S \cap T$. This is called the Inclusion-Exclusion Principle.

Inclusion-Exclusion Formula

$$n(S \cup T) = n(S) + n(T) - n(S \cap T)$$

Example 12

In the band, there are 40 brass players and 33 girls. Of these, there are 15 girls that play brass. How many members of the band play brass OR are girls?

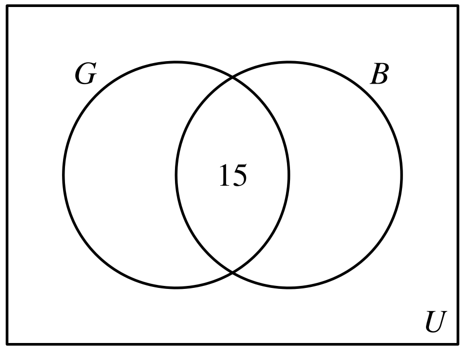
Solution: Let G be the set of girls and B be the set of brass players. From the question, we know that $n(G) = 33$ and $n(B) = 40$. Additionally, we know that 15 girls play brass. That statement can be translated as “girls and brass,” giving us the intersection set $G \cap B$ and $n(G \cap B) = 15$. The question asks for how many members play brass OR are girls. “OR” is the indication that we are looking for the union $G \cup B$.

Using the Inclusion-Exclusion Principle, we have

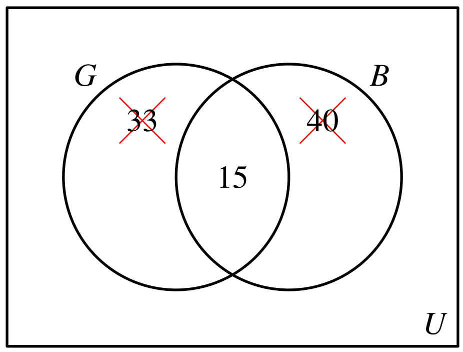
$$\begin{aligned} n(G \cup B) &= n(G) + n(B) - n(G \cap B) \\ &= 33 + 40 - 15 \\ &= 58 \end{aligned}$$

There are 58 band members who play brass or are girls. Remember that “or” in math does not mean “this one or that one only”–it means “this one, or that one, or both.”

When drawing the Venn diagram for situations like this, take special care. Start by placing the number in the intersection first.

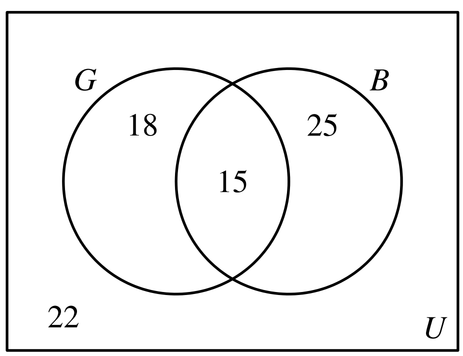


You might think you should place 33 and 40 in the sets as well. But this is wrong.



This overcounts the sets. If these numbers were correct, the diagram would be saying that there are $33 + 15 = 48$ girls and $40 + 15 = 55$ brass players in the band and that's not true.

When constructing the Venn diagram, you have to account for the numbers that have already been counted. Since there are already 15 members in set S (inside the intersection), we know that there are only $33 - 15 = 18$ more members to put in S outside the intersection. Similarly, 15 of the 40 brass players have been accounted for and that leaves $40 - 15 = 25$ brass players outside the intersection.



Adding up the numbers in the set gives $18 + 15 + 25 = 58$, the number of members that are girls or play brass. Also notice that there is a 22 sitting outside the sets but in the universal set. Since we know there are 80 total band members, there are 22 members that are "not girls" AND "do not play brass." This is the complement set $(S \cup T)'$.

Inclusion-Exclusion Formula for 3 Sets

For three sets R, S, and T:

$$\begin{aligned} n(R \cup S \cup T) &= n(R) + n(S) + n(T) & (1) \\ &\quad - n(R \cap S) - n(S \cap T) - n(T \cap R) & (2) \\ &\quad + n(R \cap S \cap T) & (3) \end{aligned}$$

Inclusion-Exclusion starts to get complicated for 3 sets or more. Let's break it down to understand it.

In line (1), we add up all of the elements in the sets R, S, and T. But this overcounts the elements in the intersections. The next step is to remove the elements that are in the intersection of two sets. This is line 2. We remove the elements in $R \cap S$, $S \cap T$, and $T \cap R$. But what happens when we do that? We accidentally removed the elements that were in the triple intersection $R \cap S \cap T$! In line 3, we add them back. This is why the technique is called Inclusion-Exclusion. You include to start, then exclude when you overcount. Then, when you exclude too much, you include them again. And round and round you go, including and then excluding, until you get to the intersection of all sets involved.

How can you see this overcount? Suppose there is an element in all three sets. How many times is this element counted in line 1? The element is counted 3 times: once for R, once for S, and once for T. We must do some excluding. How many times is the element excluded in line 2? Since the element is in $R \cap S$ and $S \cap T$ and $T \cap R$, the element is removed 3 times. We are now back down to counting the element 0 times. We must have over-excluded! In line 3, we add the element back 1 time, the correct number of times to count it.

Example 13

The director asks to see all members who play brass or is a girl or is a junior and 64 members come up. We know there are 40 brass players, 33 girls, and 18 juniors. Additionally, we know that 15 of the girls play brass, 6 of the brass players are juniors, and 9 of the juniors are girls. How many band members are junior girls that play brass?

Solution: Let G be the set of girls, B be the set of brass players, and J be the set of juniors. Then, we know the following:

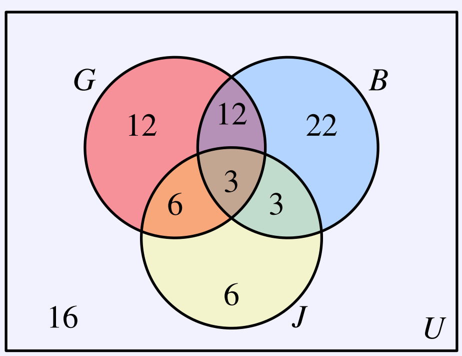
$$\begin{array}{lll} n(G) = 33 & n(B) = 40 & n(J) = 18 \\ n(G \cap B) = 15 & n(B \cap J) = 6 & n(J \cap G) = 9 \\ n(G \cup B \cup J) = 64 & n(G \cap B \cap J) = ? & \end{array}$$

Using the Extended Inclusion-Exclusion formula, we get

$$\begin{aligned} n(G \cup B \cup J) &= n(G) + n(B) + n(J) \\ &\quad - n(G \cap B) - n(B \cap J) - n(J \cap G) \\ &\quad + n(G \cap B \cap J) \\ \\ 64 &= 33 + 40 + 18 \\ &\quad - 15 - 6 - 9 \\ &\quad + n(G \cap B \cap J) \end{aligned}$$

Solving for $n(G \cap B \cap J)$, we see that there are 3 junior girls that play brass.

Here is the full filled out Venn diagram for this problem.



Practice Questions

1. At a Mexican restaurant, enchiladas are made with beef, chicken, or fajita meat. For a sauce, enchiladas come with ranchero sauce, verde sauce, or sour cream sauce. Finally, enchiladas come with corn tortillas or flour tortillas. How many different enchiladas can be made using one meat, one sauce, and one type of tortillas?

2. High schoolers must take core classes plus electives. They must take a foreign language from Spanish, French, Latin, or American Sign Language. They must sign up for an athletic activity from football, basketball, baseball/softball, soccer, or PE. Finally, they must sign up for a computer class from typing, software programming, or computer applications. How many ways can a student select their electives knowing they must have one foreign language, one athletic activity, and one computer class?

Use the following setup for the next questions.

> There are 8 red cards (numbered 1 through 8), 7 yellow cards (labeled 1 through 7), and 6 blue cards (labeled A through F) in a deck. All cards are given a different number.

3. How many ways can a red or blue card be selected?

4. How many ways can three red cards be selected in order?

5. How many ways can three red cards be selected in any order?

6. How many ways can two blue cards and three yellow cards be selected in order?

7. How many ways can two blue cards and three yellow cards be selected in any order?

8. How many ways can a hand with two cards be dealt where there is at least one "1" in the hand? [Note: If you are dealing a "hand," it is implied that the order does not matter.]

9. How many ways can a hand with three cards be dealt where the sum of the numbers on the cards is 14?

10. How many ways can a hand with five cards be dealt that contains no cards with numbers?

11. How many ways can a hand with five cards be dealt that contains all number cards?

Use the following setup for the next questions.

> A special deck of 20 cards is made of red, yellow, green, and blue colored cards. Each color has cards numbered 1 through 5.

12. How many ways can a 4-card hand contain 2 red and 2 blue cards?

13. How many ways can a 4-card hand contain all yellow cards?

14. How many ways can a 4-card hand contain no blue cards?

15. How many ways can a 4-card hand contain at least 1 blue card?

16. How many ways can a 4-card hand contain cards of all different colors?

17. How many ways can a 4-card hand contain all cards with the number "5"?

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18. The universal set U is the set of all natural numbers. Let P be the set of all prime numbers and E be the set of all event numbers. How many elements are in the set $P \cap E$?

19. The universal set U is the set of all natural numbers. Let E be the set of all even numbers. Describe E'.

20. The universal set U is the set of natural numbers less than or equal to 100. Let S be the set of multiples of 5 and T be the set of multiples of 2. How many elements are in $S \cup T$?

21. Let A be the set of all proper fractions with a denominator of 24 (reduced or not). Let B be the set of all proper fractions with a denominator of 18 (reduced or not). How many elements are in $A \cup B$?

22. In a high school with 60 students, 42 students are in Band, 19 students are in German, and 12 students are in both Band and German. How many students are in neither Band nor German?

23. There are 15 swimmers at the beach. Nine of them are in the water. Seven of them are wearing a red bathing suit. Two of them are on the beach wearing a color other than red. How many are in the water wearing red?

24. Fifty-four students in a college computer science course were asked about their knowledge in basic web design languages. They reported the following:

    - 34 know HTML
    - 33 know CSS
    - 26 know Javascript
    - 20 know HTML and CSS
    - 14 know CSS and Javascript
    - 17 know Javascript and HTML
    - 8 know HTML, CSS, and Javascript

    How many students do not know HTML, CSS, nor Javascript?

Answers: **1.** 18 **2.** 60 **3.** 14 **4.** 336 **5.** 56 **6.** 6300 **7.** 525 **8.** 39 **9.** 12 **10.** 6 **11.** 3003 **12.** 100 **13.** 5 **14.** 1365 **15.** 3480 **16.** 625 **17.** 1 **18.** 1 **19.** E' is the set of all odd numbers **20.** 60 **21.** 35 **22.** 11 **23.** 0 **24.** 4